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FORCING VERTEX TRIANGLE FREE DETOUR NUMBER OF A GRAPH

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ABSTRACT

For any vertex x in a connected graph G of order $n \geq 2$, a set $S_x \subseteq V$ is called a *x-triangle free detour set* of G if every vertex v of G lies on a $x - y$ triangle free detour for some vertex y in S_x. The *x-triangle free detour number* $dn_{\Delta f_x}(G)$ of G is the minimum order of a x-triangle free detour sets and any x-triangle free detour sets of order $dn_{\Delta f_x}(G)$ is called a $dn_{\Delta f_x}$-set of G. Let S_x be a $dn_{\Delta f_x}$-set of G. A subset $T_x \subseteq S_x$ is called an *x-forcing subset* for S_x if S_x is the unique $dn_{\Delta f_x}$-set containing T_x. An x-forcing subset for S_x of minimum order is a *minimum x-forcing subset* of S_x. The *forcing x-triangle free detour number* of S_x, denoted by $f dn_{\Delta f_x}(S_x)$, is the order of a minimum x-forcing subset for S_x. The *forcing x-triangle free detour number* of G is $f dn_{\Delta f_x}(G) = \min \{f dn_{\Delta f_x}(S_x)\}$, where the minimum is taken over all $dn_{\Delta f_x}$-sets S_x in G. We determine bounds for it and find the forcing vertex triangle free detour number of certain classes of graphs. It is shown that for every pair a, b of positive integers with $0 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G such that $f dn_{\Delta f_x}(G) = a$ and $dn_{\Delta f_x}(G) = b$..

Keywords: *triangle free detour path, vertex triangle free detour number, forcing vertex triangle free detour number*

I. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected simple graph. For basic definitions and terminologies, we refer to Chartrand et al. [1]. The concept of triangle free detour distance was introduced by Keerthi Asir and Athisayanathan [2]. A path P is called a triangle free path if no three vertices of P induce a triangle. For vertices u and v in a connected graph G, the triangle free detour distance $D_{\Delta f}(u, v)$ is the length of a longest $u - v$ triangle free path in G. A $u - v$ path of length $D_{\Delta f}(u, v)$ is called a $u - v$ triangle free detour. The concept of triangle free detour number was introduced by Sethu Ramalingam et al. [3]. A set $S \subseteq V$ is called triangle free detour set of G if every vertex of G lies on a triangle free detour joining a pair of vertices of S. The triangle free detour number $dn_{\Delta f}(G)$ of G is the minimum order of its triangle free detour sets and any triangle free detour set of order $dn_{\Delta f}(G)$ is called a triangle free detour basis of G. The concept of vertex triangle free detour number was introduced by Sethu Ramalingam et al. [4]. For any vertex x in G, a set $S_x \subseteq V$ is called a x-triangle free detour set of G if every vertex v in G lies on a $x - y$ triangle free detour in G for some vertex y in S_x. The x-triangle free detour number $dn_{\Delta f_x}(G)$ of G is the minimum order of a x-triangle free detour sets and any x-triangle free detour sets of order $dn_{\Delta f_x}(G)$ is called a $dn_{\Delta f_x}$-set of G.

The following theorems will be used in the sequel.

Theorem 1.1[4] Let x be any vertex of a connected graph G.

(i) Every extreme-vertex of G other than the vertex x (whether x is extreme-vertex or not) belong to every x-triangle free detour set.

(ii) No cut vertex of G belongs to any $dn_{\Delta f_x}$-set.

Theorem 1.2[4] Let T be a tree with t end-vertices. Then $dn_{\Delta f_x}(T) = t - 1$ or $dn_{\Delta f_x}(T) = t$ according to whether x is an end-vertex or not.

Theorem 1.3[4] Let G be the complete graph K_n of order n. For any vertex x in G, a set $S_x \subseteq V$ is a $dn_{\Delta f_x}$-set of G if and only if S_x consists of any $n - 1$ vertices of G other than x.

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Theorem 1.4[4] Let G be an even cycle of order $n \geq 4$. For any vertex x in G, a set $S_x \subseteq V$ is a $dn_{\Delta f_x}$-set of G if and only if S_x consists of exactly one vertex u of G which is adjacent to the vertex x or antipodal vertex of x.

Theorem 1.5[4] Let G be an odd cycle of order $n \geq 5$. For any vertex x in G, a set $S_x \subseteq V$ is a $dn_{\Delta f_x}$-set of G if and only if S_x consists of exactly one vertex u of G which is adjacent to the vertex x.

Theorem 1.6[4] Let G be the complete bipartite graph $K_{n,m}(1 \leq n \leq m)$. For any vertex x in G, a set $S_x \subseteq V$ is a $dn_{\Delta f_x}$-set of G if and only if S_x consists of exactly any one vertex of G other than x.

Theorem 1.7[4] Let G be a connected graph with cut-vertices and let S_x be an x-triangle free detour set of G. Then every branch at a vertex v of G contains an element of $S_x \cup \{x\}$.

Theorem 1.8[4] For any vertex x in a non-trivial connected graph G of order n, $1 \leq dn_{\Delta f_x}(G) \leq n-1$.

II. FORCING VERTEX TRIANGLE FREE DETOUR NUMBER OF A GRAPH

Definition 2.1 Let x be any vertex of a connected graph G and let S_x be a $dn_{\Delta f_x}$-set of G. A subset $T_x \subseteq S_x$ is called an x-forcing subset for S_x if S_x is the unique $dn_{\Delta f_x}$-set containing T_x. An x-forcing subset for S_x of minimum order is a minimum x-forcing subset of S_x. The forcing x-triangle free detour number of S_x, denoted by $fdn_{\Delta f_x}(S_x)$, is the order of a minimum x -forcing subset for S_x. The forcing x-triangle free detour number of G is $fdn_{\Delta f_x}(G)$ = min $\{fdn_{\Delta f_x}(S_x)\}$, where the minimum is taken over all $dn_{\Delta f_x}$-sets S_x in G.

Example 2.2 For the graph G given in Figure 2.1, the only $dn_{\Delta f_w}$-sets are $\{z,u\},\{z,y\},\{z,x\}$ of G so that $fdn_{\Delta f_w}(G) = 1$. Also the unique $dn_{\Delta f_x}$-set is $\{z,w\}$ so that $fdn_{\Delta f_x}(G) = 0$. For the graph G given in Figure 2.1(a), the only $dn_{\Delta f_w}$-sets are $\{u,v\},\{x,y\},\{u,y\},\{x,v\}$ of G so that $fdn_{\Delta f_w}(G) = 2$.

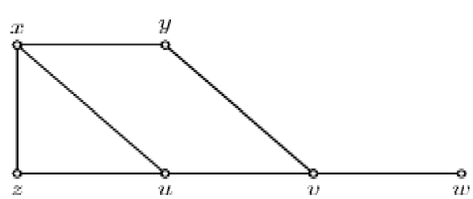

Figure 2.1 : G

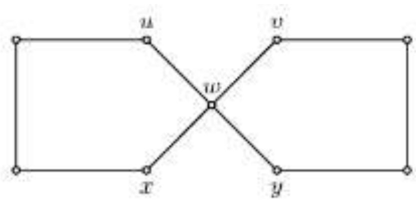

Figure 2.1 (a) : G

The following Theorem follows immediately from the definitions of vertex triangle free detour number and forcing vertex triangle free detour number of a connected graph G.

Theorem 2.3 For any vertex x in a connected graph G, $0 \leq fdn_{\Delta f_x}(G) \leq dn_{\Delta f_x}(G)$.
Proof. Let x be any vertex of G. It is clear from the definition of $fdn_{\Delta f_x}(G)$ that $fdn_{\Delta f_x}(G) \geq 0$. Let S_x be any $dn_{\Delta f_x}$-set of G. Since $fdn_{\Delta f_x}(G)$ = min $\{fdn_{\Delta f_x}(S_x)\}$, where the minimum is taken over all $dn_{\Delta f_x}$-sets S_x in G, it follows that $fdn_{\Delta f_x}(G) \leq dn_{\Delta f_x}(G)$. Thus $0 < fdn_{\Delta f_x}(G) < dn_{\Delta f_x}(G)$.

Remark 2.4 The bounds in Theorem 2.3 are sharp. For the graph G given in Figure 2.1, $fdn_{\Delta f_w}(G) = dn_{\Delta f_w}(G) = 2$. For the graph G given in Figure 2.1(b), $S_{v_1} = \{v_6\}$ is a unique $dn_{\Delta f_{v_1}}$-set of G so that $fdn_{\Delta f_{v_1}}(G) = 0$. Also, the inequalities in Theorem 2.3 can be strict. For the graph G, given in Figure 2.1, the sets $S_1 = \{w,u,z\}$, $S_2 = \{w,y,z\}$, $S_3 = \{w,x,z\}$ are a $dn_{\Delta f_v}$-sets of G so that $fdn_{\Delta f_v}(G) = 1$ and $dn_{\Delta f_v}(G) = 3$. Thus $0 \leq fdn_{\Delta f_x}(G) \leq dn_{\Delta f_x}(G)$.

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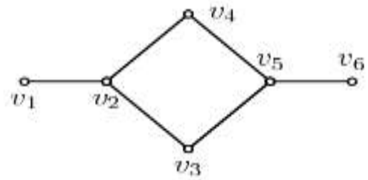

Figure 2.1(b) : G

In the following theorem we characterize graphs G for which the bounds in Therorem 2.3 are attained and also graphs for which $fdn_{\Delta f_x}(G) = 1$.

Theorem 2.5 Let x be any vertex of a connected graph G. Then

(a) $fdn_{\Delta f_x}(G) = 0$ if and only if G has a unique $dn_{\Delta f_x}$-set,
(b) $fdn_{\Delta f_x}(G) = 1$ if and only if G has at least two $dn_{\Delta f_x}$-sets, one of which is a unique $dn_{\Delta f_x}$-set containing one of its elements, and
(c) $fdn_{\Delta f_x}(G) = dn_{\Delta f_x}(G)$ if and only if no $dn_{\Delta f_x}$-set of G is the unique $dn_{\Delta f_x}$-set containing any of its proper subsets.

Proof. (a) Let $fdn_{\Delta f_x}(G) = 0$. Then, by definition, $fdn_{\Delta f_x}(S_x) = 0$ for some $dn_{\Delta f_x}$-set, S_x so that empty set φ is the minimum x-forcing subset for S_x. Since the empty set φ is a subset of every set, it follows that S_x is the unique $dn_{\Delta f_x}$-set of G. The converse is clear.

(b) Let $fdn_{\Delta f_x}(G) = 1$. Then by (a), G has at least two $dn_{\Delta f_x}$-sets. Also, since $fdn_{\Delta f_x}(G) = 1$, there is a singleton subset T of a $dn_{\Delta f_x}$-set S_x of G such that T is not a subset of any other $dn_{\Delta f_x}$-set of G. Thus S_x is the unique $dn_{\Delta f_x}$-set containing one of its elements. The converse is clear.

(c) Let $fdn_{\Delta f_x}(G) = dn_{\Delta f_x}(G)$. Then $fdn_{\Delta f_x}(S_x) = dn_{\Delta f_x}(G)$ for every $dn_{\Delta f_x}$-set S_x in G. Also by Theorem 1.8, $dn_{\Delta f_x}(G) \geq 1$ and hence $fdn_{\Delta f_x}(G) \geq 1$. Then by (a), G has at least two $dn_{\Delta f_x}$-sets and so the empty set φ is not a x-forcing subset of any $dn_{\Delta f_x}$-set of G. Since $fdn_{\Delta f_x}(S_x) = dn_{\Delta f_x}(G)$, no proper subset of S_x is an x-forcing subset of S_x. Thus no $dn_{\Delta f_x}$-set of G is the unique $dn_{\Delta f_x}$-set containing any of its proper subsets.

Conversely the data implies that G contains more than one $dn_{\Delta f_x}$-set and no subset of any $dn_{\Delta f_x}$-set S_x other than S_x is an x-forcing subset for S_x. Hence it follows that $fdn_{\Delta f_x}(G) = dn_{\Delta f_x}(G)$.

Definition 2.6 A vertex v of a connected graph G is said to be a x-triangle free detour vertex of G if v belongs to every minimum x-triangle free detour set of G.

We observe that if G has a unique $dn_{\Delta f_x}$-set S_x, then every vertex in S_x is an x -triangle free detour vertex.

Example 2.7 By Theorem 1.1(i), every extreme-vertex $v \neq x$ of any graph G is an x -triangle free detour vertex. On the other hand, there are x -triangle free detour vertices in a graph that are not end-vertices. For the graph G given in Figure 2.2, it is easily seen that $\{s\}$ is the unique minimum x-triangle free detour set of G so that the vertex s is the x -triangle free detour vertex of G but s is not an end-vertex of G.

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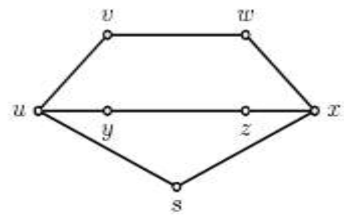

Figure 2.2 : G

Theorem 2.8 Let x be any vertex of a connected graph G and let $\digamma$ be the set of relative complements of the minimum x-forcing subsets in their respective minimum x -triangle free detour sets in G. Then $\cap_{F\in \digamma} F$ is the set of x-triangle free detour vertices of G.

Proof. Let W be the set of x-triangle free detour vertices of G. We claim that $W = \cap_{F\in \digamma} F$. Let $v \in W$. Then v is an x-triangle free detour vertex of G so that v belongs to every minimum x-triangle free detour set S_x of G. Let $T \subseteq S_x$ be any minimum x-forcing subset for any minimum x-triangle free detour set S_x of G. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S_x is the unique $dn_{\Delta f_x}$-set containing T' so that T' is an x-forcing subset for S_x with $|T'| < |T|$, which is a contradiction to T a minimum x-forcing subset for S_x. Thus $v \notin T$ and so $v \in F$, where F is the relative complement of T in S_x. Hence $v \in \cap_{F\in \digamma} F$ so that $W \subseteq \cap_{F\in \digamma} F$.

Conversely, let $v \in \cap_{F\in \digamma} F$. Then v belongs to the relative complement of T in S_x for every T and every S_x such that $T \subseteq S_x$, where T is a minimum x-forcing subset for S_x. Since F is the relative complement of T in S_x, $F \subseteq S_x$ and so $v \in S_x$ for every S_x so that v is an x-triangle free detour vertex of G. Thus $v \in W$ and so $\cap_{F\in \digamma} F \subseteq W$. Hence $W = \cap_{F\in \digamma} F$.

Theorem 2.9 Let x be any vertex of a connected graph G and let S_x be any $dn_{\Delta f_x}$-set of G.

(i) No cut vertex of G belongs to any minimum x-forcing subset of S_x.
(ii) No x-triangle free detour vertex of G belongs to any minimum x-forcing subset of S_x.

Proof. Let x be any vertex of a connected graph G and let S_x be any minimum x-triangle free detour set of G.

(i) Since any minimum x-forcing subset of S_x is a subset of S_x, the result follows from Theorem 1.1 (ii).
(ii) The proof is contained in the proof of the first part of Theorem 2.8.

Corollary 2.10 Let x be any vertex of a connected graph G. If G contains k end-vertices, then $fdn_{\Delta f_x}(G) \leq dn_{\Delta f_x}(G) - k + 1$.

Proof. This follows from Theorem 1.1(i) and Theorem 2.9(ii).

Remark 2.11 The bounds in Corollary 2.10 are sharp. For a tree T with k end-vertices, $fdn_{\Delta f_x}(T) = dn_{\Delta f_x}(T) - k + 1$ for any end-vertex x in T.

Theorem 2.12 Let G be a connected graph of order n.

(a) If G is a tree with t end-vertices, then $fdn_{\Delta f_x}(G) = 0$ for every vertex x in T.
(b) If G is the complete bipartite graph $K_{n,m}$, then $fdn_{\Delta f_x}(G) = 1$ for every vertex x in G.
(c) If G is the complete graph K_n, then $fdn_{\Delta f_x}(G) = 0$ for every vertex x in G.
(d) If G is the cycle $C_n (n \geq 4)$, then $fdn_{\Delta f_x}(G) = 1$ for every vertex x in G.

Proof. (a) By Theorem 1.2, $dn_{\Delta f_x}(G) = t - 1$ or $dn_{\Delta f_x}(G) = t$ according to whether x is an end-vertex or not. Since the set of all end-vertices of a tree is the unique $dn_{\Delta f_x}$-set, the result follows from Theorem 2.5(a).

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(b) By Theorem 1.6, a set S_x consists of exactly any one vertex of G For the vertex v in G there are two or more vertices adjacent with v. Thus the vertex v belongs to x-triangle free detour basis of G. Thus the result follows.

(c) By Theorem 1.3, a set S_x consists of any $n-1$ vertices of G. Also the set of all $n-1$ vertices of G is the unique unique $dn_{\Delta f_x}$-set, the result follows from Theorem 2.5.

(d) By Therem 1.4 or 1.5 (according as G is even or odd), a set S_x consists of one vertex which is adjacent to x or antipodal vertex of x. For each vertex v in G there are two vertices adjacent with v. Thus the vertex v belongs to exactly one $dn_{\Delta f_x}$-set of G. Hence it follows that a set consisting of a single vertex is a forcing subset for any $dn_{\Delta f_x}$-set of G. Thus the result follows.

The following theorem gives a realization result.
Theorem 2.13 For each pair a, b of integers with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph G such that $fdn_{\Delta f_x}(G) = a$ and $dn_{\Delta f_x}(G) = b$ for some vertex x in G.

Proof. We consider two cases, according to whether $a = 0$ or $a \geq 1$.

Case 1. Let $a = 0$. Let G be any tree with $b + 1$ end vertices. Then for any end vertex x in G, $fdn_{\Delta f_x}(G) = 0$ by Theorem 2.9(i) and by Theorem 2.5(a).

Case 2. Let $a \geq 1$. For each integer i with $1 \leq i \leq a$, let F_i be a copy of K_2, where v_i and v_i' are the vertices of F_i. Let $K_{1,b-2a}$ be the star at x and let $U = \{u_1, u_2, \ldots, u_{b-2a}\}$ be the set of end vertices of $K_{1,b-2a}$. Let G be the graph obtained by joining the vertex x with the vertices of $F_1, F_2, \ldots, F_a$. The graph G is shown in Figure 2.3.

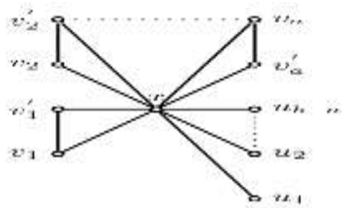

Figure 2.3 : G

First, we show that $dn_{\Delta f_x}(G) = b$ for some vertex x in G. By Theorem 1.1 and Theorem 1.7, every $dn_{\Delta f_x}$-set of G contains U and every vertex from each $F_i (1 \leq i \leq a)$. Thus $dn_{\Delta f_x}(G) \geq (b - 2a) + 2a = b$. Let $S_x = U \cup \{v_1, v_2, \ldots, v_a\}$. It is clear that S_x is an $dn_{\Delta f_x}$-set of G and so $dn_{\Delta f_x}(G) \leq |S_x| = (b - 2a) + 2a = b$. Thus $dn_{\Delta f_x}(G) = b$. Next we show that $fdn_{\Delta f_x}(G) = a$. Since $fdn_{\Delta f_x}(G) = b$, we observe that every $dn_{\Delta f_x}$-set of G contains $U$and exactly one vertex from each $F_i (1 \leq i \leq a)$. Let $T \subseteq S_x$ be any minimum x-forcing subset of S_x. Then $T \subseteq S_x - U$, by Theorem 2.9 (ii) and so $|T| \leq a$. If $|T| < a$, then there is a vertex v_i of $F_i(1 \leq i \leq a$ such that $v_i \in S_x$ and $v_i \notin T$. Let v_i' be the other vertex of F_i. Then $S_x' = (S_x - v_i) \cup v_i'$ is a $dn_{\Delta f_x}$-set of G different from S_x such that it contains T, which is a contradiction to T is a minimum x-forcing subset of S_x. Thus $fdn_{\Delta f_x}(G) = a$.

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